Logic and Implication

An introduction to the general algebraic study of non-classical logics

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SLALM 2019

Outline

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- Protoalgebraic logics and their subclasses are based on a general notion of equivalence.
- Implication has a crucial role in reasoning (entailment, consequence, preservation of truth,...)
- The goal of this course is to present an AAL theory based on implication, together with a wealth of examples of (non-)classical logics.

Outline

Basic syntactical notions – 1

Propositional language: a countable type $\mathcal{L}$, i.e. a function $ar\colon C_{\mathcal{L}} \to \mathrm{N}$, where $C_{\mathcal{L}}$ is a countable set of symbols called connectives, giving for each one its arity. Nullary connectives are also called truth-constants. We write $\langle c, n\rangle \in \mathcal{L}$ whenever $c \in C_{\mathcal{L}}$ and $ar(c) = n$.

Formulas: Let Var be a fixed infinite countable set of symbols called variables. The set $Fm_{\mathcal{L}}$ of formulas in $\mathcal{L}$ is the least set containing Var and closed under connectives of $\mathcal{L}$, i.e. for each $\langle c, n\rangle \in \mathcal{L}$ and every $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}$, $c(\varphi_1, \dots, \varphi_n)$ is a formula.

Substitution: a mapping $\sigma\colon Fm_{\mathcal{L}} \to Fm_{\mathcal{L}}$, such that $\sigma(c(\varphi_1, \dots, \varphi_n)) = c(\sigma(\varphi_1), \dots, \sigma(\varphi_n))$ holds for each $\langle c, n\rangle \in \mathcal{L}$ and every $\varphi_1, \dots, \varphi_n \in Fm_{\mathcal{L}}$.

Consecution: a pair $\Gamma \rhd \varphi$, where $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$.

Basic syntactical notions – 2

A set L of consecutions can be seen as a relation between sets of formulas and formulas. We write '$\Gamma \vdash_{\mathrm{L}} \varphi$' instead of '$\Gamma \rhd \varphi \in \mathrm{L}$'.

Definition

A set L of consecutions in $\mathcal{L}$ is called a logic in $\mathcal{L}$ whenever

- If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathrm{L}} \varphi$. (Reflexivity)
- If $\Delta \vdash_{\mathrm{L}} \psi$ for each $\psi \in \Gamma$ and $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Cut)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each substitution σ. (Structurality)

Observe that reflexivity and cut entail:

- If $\Gamma \vdash_{\mathrm{L}} \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash_{\mathrm{L}} \varphi$. (Monotonicity)

The least logic Min is described as:

$$\Gamma \vdash_{\mathrm{Min}} \varphi \qquad \text{iff} \qquad \varphi \in \Gamma.$$

Basic syntactical notions – 3

Theorem: a consequence of the empty set
(note that Min has no theorems).

Inconsistent logic Inc: the set of all consecutions
(equivalently: a logic where all formulas are theorems).

Almost Inconsistent logic AInc: the maximum logic without theorems (note that $\Gamma, \varphi \vdash_{\mathrm{AInc}} \psi$).

Theory: a set of formulas T such that if $T \vdash_{\mathrm{L}} \varphi$ then $\varphi \in T$. We denote by $\mathrm{Th}(\mathrm{L})$ the set of all theories of L.

Note that

- $\mathrm{Th}(\mathrm{L})$ can be seen as a closure system. We denote by $\mathrm{Th}_{\mathrm{L}}(\Gamma)$ the theory generated in $\mathrm{Th}(\mathrm{L})$ by Γ (i.e., the intersection of all theories containing Γ).
- $\mathrm{Th}_{\mathrm{L}}(\Gamma) = \{\varphi \in Fm_{\mathcal{L}} \mid \Gamma \vdash_{\mathrm{L}} \varphi\}$.
- The set of all theorems is the least theory and it is generated by the empty set.

Basic syntactical notions – 4

Axiomatic system: a set $\mathcal{AS}$ of consecutions closed under substitutions. An element $\Gamma \rhd \varphi$ is an

- axiom if $\Gamma = \emptyset$,
- finitary deduction rule if Γ is a finite,
- infinitary deduction rule otherwise.

An axiomatic system is finitary if all its rules are finitary.

Proof: a proof of a formula φ from a set of formulas Γ in $\mathcal{AS}$ is a well-founded rooted tree labeled by formulas such that

- its root is labeled by φ and leaves by axioms of $\mathcal{AS}$ or elements of Γ and
- if a node is labeled by ψ and $\Delta \neq \emptyset$ is the set of labels of its preceding nodes, then $\Delta \rhd \psi \in \mathcal{AS}$.

We write $\Gamma \vdash_{\mathcal{AS}} \varphi$ if there is a proof of φ from Γ in $\mathcal{AS}$.

Basic syntactical notions – 5

Lemma

$\vdash_{\mathcal{AS}}$ *is the least logic containing the axiomatic system* $\mathcal{AS}$.

Presentation: We say that $\mathcal{AS}$ is an axiomatic system for (or a presentation of) the logic L if $\mathrm{L} = \vdash_{\mathcal{AS}}$. A logic is said to be finitary if it has some finitary presentation.

Lemma

A logic L *is finitary iff for each set of formulas* $\Gamma \cup \{\varphi\}$ *we have: if* $\Gamma \vdash_{\mathrm{L}} \varphi$, *then there is a finite* $\Gamma' \subseteq \Gamma$ *such that* $\Gamma' \vdash_{\mathrm{L}} \varphi$.

Note that $\mathrm{Inc}, \mathrm{AInc}, \mathrm{Min}$ are finitary because:

Inc	is axiomatized by	axioms $\{\varphi \mid \varphi \in Fm_{\mathcal{L}}\}$
AInc	is axiomatized by	unary rules $\{\varphi \rhd \psi \mid \varphi, \psi \in Fm_{\mathcal{L}}\}$
Min	is axiomatized by	by the empty set

More interesting examples

Finitary axiomatic system for BCI in $\mathcal{L}_{\rightarrow} = \{\rightarrow\}$

B $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$

C $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$

I $\varphi \rightarrow \varphi$

MP $\varphi, \varphi \rightarrow \psi \rhd \psi$

Finitary axiomatic system for BCK in $\mathcal{L}_{\rightarrow} = \{\rightarrow\}$

B $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$

C $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$

K $\varphi \rightarrow (\psi \rightarrow \varphi)$

MP $\varphi, \varphi \rightarrow \psi \rhd \psi$

Even more interesting examples

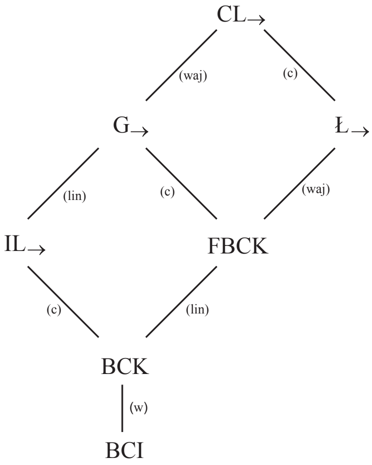
Consider the following axioms in $\mathcal{L}_{\to}$:

(c)	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	contraction
(waj)	$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$	Wajsberg axiom
(lin)	$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$	linearity

and define the following logics:

Logic	Presentation
FBCK	BCK extended by (lin)
$\mathrm{IL}_{\to}$	BCK extended by (c)
$\mathrm{G}_{\to}$	BCK extended by (c) and (lin)
$\text{Ł}_{\to}$	BCK extended by (lin) and (waj)
$\mathrm{CL}_{\to}$	BCK extended by (c), (lin), and (waj)

Prominent axiomatic extensions of BCI



Famous examples

$\mathcal{L}_{\mathrm{CL}} = \{\to, \wedge, \vee, \overline{0}\}$.

IL (intuitionistic logic): axiomatic expansion of $\mathrm{IL}_{\to}$
CL (classical logic): axiomatic expansion of $\mathrm{CL}_{\to}$
$\mathrm{Ł}$ (Łukasiewicz logic): axiomatic expansion of $\mathrm{Ł}_{\to}$
G (Gödel–Dummett logic): axiomatic expansion of $\mathrm{G}_{\to}$

by the axioms:

$(\bot)$	$\overline{0} \to \varphi$
(axAdj)	$\varphi \to (\psi \to \varphi \wedge \psi)$
(LB_1)	$\varphi \wedge \psi \to \varphi$
(LB_2)	$\varphi \wedge \psi \to \psi$
(axInf)	$(\varphi \to \psi) \wedge (\varphi \to \chi) \to (\varphi \to \psi \wedge \chi)$
(UB_1)	$\varphi \to \varphi \vee \psi$
(UB_2)	$\psi \to \varphi \vee \psi$
(axSup)	$(\varphi \to \chi) \wedge (\psi \to \chi) \to (\varphi \vee \psi \to \chi)$

Remarks on the famous examples – 1

- Classical logic has numerous other presentations more common than the one used here.
- Gödel–Dummett is usually presented in a language where $\vee$ is a defined connective:

$$\varphi \vee \psi = ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$$

Then, Gödel–Dummett logic is the axiomatic extension of IL by the axiom of *prelinearity*:

$$(\varphi \to \psi) \vee (\psi \to \varphi)$$

Remarks on the famous examples – 2

- Łukasiewicz logic is usually presented in a language where $\wedge$ and $\vee$ are defined connectives:

$$\varphi \vee \psi = (\varphi \to \psi) \to \psi \qquad\qquad \varphi \wedge \psi = \neg(\neg\varphi \vee \neg\psi)$$

  with an axiomatic system consisting of *modus ponens* and axioms (B), (K), (waj), and

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi).$$

  Also, the following two connectives are usually defined in Łukasiewicz logic:

$$\varphi \mathbin{\&} \psi = \neg(\varphi \to \neg\psi) \qquad\qquad \varphi \oplus \psi = \neg\varphi \to \psi.$$

An infinitary example

A prominent extension of Łukasiewicz logic, denoted as Ł_∞, is obtained by adding the following infinitary rule:

$$\{\neg\varphi \to \varphi \,\&\, \overset{n}{\ldots} \,\&\, \varphi \mid n \geq 1\} \rhd \varphi$$

Basic semantical notions – 1

$\mathcal{L}$-algebra: $\boldsymbol{A} = \langle A, \langle c^{\boldsymbol{A}} \mid c \in C_{\mathcal{L}} \rangle\rangle$, where $A \neq \emptyset$ (universe) and $c^{\boldsymbol{A}} \colon A^n \to A$ for each $\langle c, n\rangle \in \mathcal{L}$.

Algebra of formulas: the algebra $\boldsymbol{Fm}_{\mathcal{L}}$ with domain $Fm_{\mathcal{L}}$ and operations $c^{\boldsymbol{Fm}_{\mathcal{L}}}$ for each $\langle c, n\rangle \in \mathcal{L}$ defined as:

$$c^{\boldsymbol{Fm}_{\mathcal{L}}}(\varphi_1, \ldots, \varphi_n) = c(\varphi_1, \ldots, \varphi_n).$$

$\boldsymbol{Fm}_{\mathcal{L}}$ is the absolutely free algebra in language $\mathcal{L}$ with generators Var.

Homomorphism of algebras: a mapping $f \colon A \to B$ such that for every $\langle c, n\rangle \in \mathcal{L}$ and every $a_1, \ldots, a_n \in A$,

$$f(c^{\boldsymbol{A}}(a_1, \ldots, a_n)) = c^{\boldsymbol{B}}(f(a_1), \ldots, f(a_n)).$$

Note that substitutions are exactly endomorphisms of $\boldsymbol{Fm}_{\mathcal{L}}$.

Examples of algebras – 1

Boolean algebra: $\boldsymbol{A} = \langle A, \wedge, \vee, \neg, \overline{0}, \overline{1} \rangle$, where $\langle A, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is a bounded distributive lattice and for every $a \in A$:

$$a \wedge \neg a = \overline{0} \text{ and } a \vee \neg a = \overline{1} \qquad \text{(\textit{complement})}$$

Prototypical example: power set algebra of a set A, i.e. the structure $\langle P(A), \cap, \cup, -, \emptyset, A \rangle$, where for every $X \subseteq A$ we have $-X = A \setminus X$.

Stone's representation theorem: each Boolean algebra can be embedded into a Boolean algebra defined over the power set algebra of some set.

We denote the class of all Boolean algebras as $\mathbb{BA}$.

Examples of algebras – 2

Heyting algebra: $\boldsymbol{A} = \langle A, \wedge, \vee, \rightarrow, \overline{0}, \overline{1}\rangle$, where $\boldsymbol{A} = \langle A, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is a bounded distributive lattice and for every $a, b, c \in A$:

$$a \wedge b \leq c \text{ if, and only, if } a \leq b \rightarrow c \qquad (\textit{residuation})$$

where $\leq$ is the canonical lattice order.

$\rightarrow$ is called the residuum of $\wedge$.

Pseudocomplement: $\neg a = a \rightarrow \overline{0}$ for $a \in A$.

We denote the class of all Heyting algebras as $\mathbb{HA}$.

Each Boolean algebra can be seen as a Heyting algebra where the residuum is defined as $a \rightarrow b = \neg a \vee b$. Therefore, Boolean algebras turn out to be exactly the Heyting algebras in which $\neg$ satisfies the complement condition.

Examples of algebras – 3

Gödel algebra or G-algebra: A Heyting algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \rightarrow, \overline{0}, \overline{1}\rangle$ such that for every $a, b \in A$:

$$(a \rightarrow b) \vee (b \rightarrow a) = \overline{1}. \qquad \textit{(prelinearity)}$$

We denote the class of all G-algebras as $\mathbb{G}$.

$\mathbb{BA} \subseteq \mathbb{G} \subseteq \mathbb{HA}$

Examples of algebras – 4

MV-algebra: $\langle A, \oplus, \neg, \overline{0} \rangle$, where $\oplus$ is a binary operation, $\neg$ is a unary operation and $\overline{0}$ is a constant such that the following are satisfied for any $a, b, c \in A$:

1. $a \oplus (b \oplus c) = (a \oplus b) \oplus c$
2. $a \oplus b = b \oplus a$
3. $a \oplus \overline{0} = \overline{0}$
4. $\neg\neg a = a$,
5. $\neg\overline{0} \oplus a = \neg\overline{0}$,
6. $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$.

We denote the class of all MV-algebras as $\mathbb{MV}$.

Lattice operations

Proposition

Let $\langle A, \oplus, \neg, \overline{0} \rangle$ *be an* MV*-algebra. For each* $a, b \in A$ *we define:*

- $a \,\&\, b = \neg(\neg a \oplus \neg b)$
- $a \to b = \neg(a \,\&\, \neg b)$
- $\overline{1} = \neg\overline{0}$
- $a \vee b = a \oplus (b \,\&\, \neg a)$
- $a \wedge b = a \,\&\, (b \oplus \neg a)$

  Then:

  1. $\langle A, \wedge, \vee, \overline{0}, \overline{1} \rangle$ *is a bounded distributive lattice, and*
  2. *for each* $a, b \in A$*, we have:* $a \,\&\, b \leq c$ *iff* $a \leq b \to c$*.*

Examples of algebras – 5

- the standard G-algebra: $[0,1]_{\mathrm{G}} = \langle [0,1], \wedge, \vee, \rightarrow, 0, 1\rangle$, where $\wedge$ and $\vee$ are the lattice operations given by the natural order in $[0,1]$, and for each $a, b \in [0,1]$:

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

- the standard MV-algebra: $[0,1]_{\text{Ł}} = \langle [0,1], \oplus, \neg, 0\rangle$, where for each $a, b \in [0,1]$, $a \oplus b = \min\{a+b, 1\}$ and $\neg a = 1 - a$. The lattice operations defined in the previous proposition coincide with the lattice operations given by the natural order in $[0,1]$.

Basic semantical notions – 2

$\mathcal{L}$-matrix: a pair $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra called the algebraic reduct of $\mathbf{A}$, and F is a subset of A called the filter of $\mathbf{A}$. The elements of F are called designated elements of $\mathbf{A}$.

A matrix $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ is

- trivial if $F = A$.
- finite if A is finite.
- Lindenbaum if $\boldsymbol{A} = \boldsymbol{Fm}_{\mathcal{L}}$.

$\boldsymbol{A}$-evaluation: a homomorphism from $\boldsymbol{Fm}_{\mathcal{L}}$ to $\boldsymbol{A}$, i.e. a mapping $e\colon Fm_{\mathcal{L}} \to A$, such that for each $\langle c, n \rangle \in \mathcal{L}$ and each n-tuple of formulas $\varphi_1, \ldots, \varphi_n$ we have:

$$e(c(\varphi_1, \ldots, \varphi_n)) = c^{\boldsymbol{A}}(e(\varphi_1), \ldots, e(\varphi_n)).$$

Basic semantical notions – 3

Semantical consequence: A formula φ is a semantical consequence of a set Γ of formulas w.r.t. a class $\mathbb{K}$ of $\mathcal{L}$-matrices if for each $\langle \boldsymbol{A}, F\rangle \in \mathbb{K}$ and each $\boldsymbol{A}$-evaluation e, we have $e(\varphi) \in F$ whenever $e[\Gamma] \subseteq F$; we denote it by $\Gamma \models_{\mathbb{K}} \varphi$.

Exercise 1

Let $\mathbb{K}$ be a class of $\mathcal{L}$-matrices. Then $\models_{\mathbb{K}}$ is a logic in $\mathcal{L}$.

Lemma (Tabular logics)

Furthermore, if $\mathbb{K}$ is a finite class of finite matrices, then the logic $\models_{\mathbb{K}}$ is finitary.

L-matrix: Let L be a logic in $\mathcal{L}$ and $\mathbf{A}$ an $\mathcal{L}$-matrix. We say that $\mathbf{A}$ is an L-matrix if $\mathrm{L} \subseteq \models_{\mathbf{A}}$. We denote the class of L-matrices by $\mathbf{MOD}(\mathrm{L})$.

Basic semantical notions – 4

Lemma (Images and preimages of models)

Let L *be a logic in* $\mathcal{L}$ *and a mapping* $g\colon \boldsymbol{A} \to \boldsymbol{B}$ *be a homomorphism of* $\mathcal{L}$*-algebras* $\boldsymbol{A}, \boldsymbol{B}$*. Then:*

- $\langle \boldsymbol{A}, g^{-1}[G] \rangle \in \mathbf{MOD}(\mathrm{L})$, *whenever* $\langle \boldsymbol{B}, G \rangle \in \mathbf{MOD}(\mathrm{L})$.
- $\langle \boldsymbol{B}, g[F] \rangle \in \mathbf{MOD}(\mathrm{L})$, *whenever* $\langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}(\mathrm{L})$ *and* g *is surjective and* $g(x) \in g[F]$ *implies* $x \in F$.

Basic semantical notions – 5

Logical filter: Given a logic L in $\mathcal{L}$ and an $\mathcal{L}$-algebra $\boldsymbol{A}$, a subset $F \subseteq A$ is an L-*filter* if $\langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$. We denote by $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ the set of all L-filters over $\boldsymbol{A}$.

$\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ is a closure system and can be given a lattice structure by defining for any $F, G \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$, $F \wedge G = F \cap G$ and $F \vee G = \mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(F \cup G)$.

Generated filter: Given a set $X \subseteq A$, the logical filter generated by X is $\mathrm{Fi}_{\mathrm{L}}^{\boldsymbol{A}}(X) = \bigcap\{F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A}) \mid X \subseteq F\}$.

$$\mathcal{F}i_{\mathrm{Min}}(\boldsymbol{A}) = \mathcal{P}(A) \qquad \mathcal{F}i_{\mathrm{AInc}}(\boldsymbol{A}) = \{\emptyset, A\} \qquad \mathcal{F}i_{\mathrm{Inc}}(\boldsymbol{A}) = \{A\}$$

Examples of logical filters – 1

Exercise 2

- Let $\boldsymbol{A}$ be a Heyting algebra. Then $F \in \mathcal{F}i_{\mathrm{IL}}(\boldsymbol{A})$ iff F is a lattice filter on $\boldsymbol{A}$.
- Let $\boldsymbol{A}$ be a G-algebra. Then $F \in \mathcal{F}i_{\mathrm{G}}(\boldsymbol{A})$ iff F is a lattice filter on $\boldsymbol{A}$.
- Let $\boldsymbol{A}$ be a Boolean algebra. Then $F \in \mathcal{F}i_{\mathrm{CL}}(\boldsymbol{A})$ iff F is a lattice filter on $\boldsymbol{A}$.
- Let $\boldsymbol{A}$ be an MV-algebra. Then $F \in \mathcal{F}i_{\mathrm{Ł}}(\boldsymbol{A})$ iff F is a lattice filter on $\boldsymbol{A}$ and for each $x, y \in A$ such that $x, x \to y \in F$ we have $y \in F$.

Examples of logical filters – 2

$\mathbf{A} = \langle [0,1]_{\mathrm{G}}, (0,1] \rangle \in \mathbf{MOD}(\mathrm{CL})$.

Indeed, it is a model of IL and it is easy to check that:

- $\models_{\mathbf{A}} ((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$
- $\models_{\mathbf{A}} ((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$

Examples of logical filters – 3

Now we can show that $Ł_\infty$ is not finitary (hence, a proper extension of Łukasiewicz logic).

$\mathbf{M}_Ł = \langle [0,1]_Ł, \{1\} \rangle \in \mathbf{MOD}(Ł_\infty)$. However, for each positive $k \in$ N

$$\{\neg\varphi \to \varphi^n \mid 1 \leq n < k\} \not\models_{\mathbf{M}_Ł} \varphi,$$

where by φ^n we denote $\varphi \,\&\, \overset{n}{\ldots} \,\&\, \varphi$. Indeed, it suffices to take the evaluation $e(\varphi) = \frac{k}{k+1}$ and note that $e(\varphi)^n = \frac{k-n}{k+1} \geq \frac{1}{k+1} = e(\neg\varphi)$ for $n < k$

Examples of logical filters – 4

A model of Ł which is not a model of $Ł_\infty$.

$\boldsymbol{C} = \langle C, \oplus, \neg, \overline{0} \rangle$ (Chang algebra):

- $C = \{\langle 0, i\rangle \mid i \in \mathrm{N}\} \cup \{\langle 1, -i\rangle \mid i \in \mathrm{N}\}$
- $\overline{0} = \langle 0, 0\rangle$
- $\langle x, i\rangle \oplus \langle y, j\rangle = \begin{cases} \langle 1, 0\rangle & \text{if } x + y = 2 \\ \langle 1, 0\rangle & \text{if } x + y = 1 \text{ and } i + j \geq 0 \\ \langle x + y, i + j\rangle & \text{otherwise} \end{cases}$
- $\neg\langle x, i\rangle = \langle 1 - x, -i\rangle$.

$$\langle x, i\rangle \,\&\, \langle y, j\rangle = \begin{cases} \langle 0, 0\rangle & \text{if } x + y = 0 \\ \langle 0, 0\rangle & \text{if } x + y = 1 \text{ and } i + j \leq 0 \\ \langle x + y - 1, i + j\rangle & \text{otherwise} \end{cases}$$

Now we consider the matrix $\mathbf{C} = \langle \boldsymbol{C}, \{\langle 1, 0\rangle\}\rangle$ and show that

$$\{\neg\varphi \to \varphi^n \mid n \geq 1\} \not\models_{\mathbf{C}} \varphi.$$

Indeed, $e(\varphi) = \langle 1, -1\rangle$, and compute by induction that $\langle 1, -1\rangle^n = \langle 1, -n\rangle$ and so $e(\neg\varphi \to \varphi^n) = \langle 1, -1\rangle \oplus \langle 1, -n\rangle = \langle 1, 0\rangle$.

Examples of logical filters – 5

For each $n \geq 2$, take the subalgebra $\boldsymbol{MV_n}$ of $[0,1]_{\text{Ł}}$ with the n-element domain $\{0, \frac{1}{n-1}, \ldots, 1\}$ and the matrix $\mathbf{Ł}_n = \langle \boldsymbol{MV_n}, \{1\} \rangle$.

$\models_{\mathbf{Ł}_n}$ is a finitary logic (by the lemma on tabular logics).

$\mathbf{Ł}_n \in \mathbf{MOD}(\text{Ł})$ (by the lemma on preimages of models).

$\mathbf{Ł}_n \in \mathbf{MOD}(\text{Ł}_\infty)$ (checking the semantical validity of the infinitary rule).

$\text{Ł}_\infty \subsetneq \models_{\{\mathbf{Ł}_n \mid n \geq 2\}}$

The rule $\{(p_i \to p_{i+1})^{i(i+1)} \to q \mid i > 0\} \rhd q$ can be checked to be sound in each $\mathbf{Ł}_n$, while the evaluation $e(q) = 0$ and $e(p_i) = \frac{1}{i}$ shows that it is not derivable in Ł_∞.

Examples of logical filters – 6

Exercise 3

The logic BCI: By $\boldsymbol{M}$ we denote the $\mathcal{L}_{\to}$-algebra with domain $\{\bot, \top, t, f\}$ and:

$\to^{\boldsymbol{M}}$	$\top$	t	f	$\bot$
$\top$	$\top$	$\bot$	$\bot$	$\bot$
t	$\top$	t	f	$\bot$
f	$\top$	$\bot$	t	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$

Check that

$$\mathcal{F}i_{\mathrm{BCI}}(\boldsymbol{M}) = \{\{t, \top\}, \{t, f, \top\}, M\}.$$

The first completeness theorem

Proposition

For any logic L *in a language* $\mathcal{L}$, $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{Fm}_{\mathcal{L}}) = \mathrm{Th}(\mathrm{L})$.

Theorem

Let L *be a logic. Then for each set* Γ *of formulas and each formula* φ *the following holds:* $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\Gamma \models_{\mathbf{MOD}(\mathrm{L})} \varphi$.

Outline

Completeness theorem for classical logic

- Suppose that $T \in \mathrm{Th}(\mathrm{CL})$ and $\varphi \notin T$ ($T \nvdash_{\mathrm{CL}} \varphi$). We want to show that $T \not\models \varphi$ in some meaningful semantics.
- $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}, T \rangle} \varphi$. 1st completeness theorem
- $\langle \alpha, \beta \rangle \in \Omega(T)$ iff $\alpha \leftrightarrow \beta \in T$ (congruence relation on $\boldsymbol{Fm}_{\mathcal{L}}$ compatible with T: if $\alpha \in T$ and $\langle \alpha, \beta \rangle \in \Omega(T)$, then $\beta \in T$).
- Lindenbaum–Tarski algebra: $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T)$ is a Boolean algebra and $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}/\Omega(T), T/\Omega(T) \rangle} \varphi$.
  2nd completeness theorem
- Lindenbaum Lemma: If $\varphi \notin T$, then there is a maximal consistent $T' \in \mathrm{Th}(\mathrm{CL})$ such that $T \subseteq T'$ and $\varphi \notin T'$.
- $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T') \cong \boldsymbol{2}$ (subdirectly irreducible Boolean algebra) and $T \not\models_{\langle \boldsymbol{2}, \{1\} \rangle} \varphi$. 3rd completeness theorem

Weakly implicative logics

Definition

A logic L in a language $\mathcal{L}$ is weakly implicative if there is a binary connective $\to$ (primitive or definable) such that:

$$
\begin{array}{ll}
(\mathrm{R}) & \vdash_{\mathrm{L}} \varphi \to \varphi \\
(\mathrm{MP}) & \varphi, \varphi \to \psi \vdash_{\mathrm{L}} \psi \\
(\mathrm{T}) & \varphi \to \psi, \psi \to \chi \vdash_{\mathrm{L}} \varphi \to \chi \\
(\mathrm{sCng}) & \varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{L}} c(\chi_1, \dots, \chi_i, \varphi, \dots, \chi_n) \to \\
& \qquad\qquad\qquad\qquad c(\chi_1, \dots, \chi_i, \psi, \dots, \chi_n)
\end{array}
$$

for each $\langle c, n\rangle \in \mathcal{L}$ and each $0 \leq i < n$.

Examples of (non-)weakly implicative logics – 1

- Min and AInc are not weakly implicative because they have no theorems (and hence no connective can satisfy the reflexivity requirement).
- Inc is weakly implicative (any binary connective works).
- Since prefixing is a theorem of BCI, in particular we obtain

  $$\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{BCI}} (\varphi \to \chi) \to (\psi \to \chi)$$
  $$\varphi \to \psi, \psi \to \varphi \vdash_{\mathrm{BCI}} (\chi \to \varphi) \to (\chi \to \psi)$$

  Thus all extensions of BCI are weakly implicative.

Examples of (non-)weakly implicative logics – 2

The axiomatic expansions of BCK we have seen are weakly implicative.
It is enough to show:

$$\varphi \to \psi, \psi \to \varphi \vdash \varphi \vee \chi \to \psi \vee \chi$$
$$\varphi \to \psi, \psi \to \varphi \vdash \varphi \wedge \chi \to \psi \wedge \chi$$
$$\varphi \to \psi, \psi \to \varphi \vdash \chi \vee \varphi \to \chi \vee \psi$$
$$\varphi \to \psi, \psi \to \varphi \vdash \chi \wedge \varphi \to \chi \wedge \psi$$

Observe that the equivalence connective $\equiv$ (defined as $(\varphi \to \psi) \wedge (\psi \to \varphi)$) is also a weak implication, though it differs substantially from $\to$ in logical behavior, for instance we have $\varphi \vdash \psi \to \varphi$ but not $\varphi \vdash \psi \equiv \varphi$.

Modal logics – 1

$\mathcal{L}_\Box$: $\mathcal{L}_{\mathrm{CL}}$ with an additional unary connective $\Box$.

$$
\begin{array}{ll}
(\mathrm{K}_\Box) & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\
(\mathrm{T}_\Box) & \Box\varphi \to \varphi \\
(4_\Box) & \Box\varphi \to \Box\Box\varphi \\
(\mathrm{Nec}_\Box) & \varphi \rhd \Box\varphi
\end{array}
$$

Global modal logics:

- K is the expansion of CL by $(\mathrm{K}_\Box)$ and $(\mathrm{Nec}_\Box)$.
- T: axiomatic extension of K by $(\mathrm{T}_\Box)$
- $\mathrm{K4}$: axiomatic extension of K by $(4_\Box)$
- $\mathrm{S4}$: axiomatic extension of T by $(4_\Box)$

Modal logics – 2

Local modal logics:

If L is a global modal logic, its local variant can be defined in two equivalent ways:

1. as the axiomatic expansion of CL by all the theorems of L,
2. by taking as axioms all the formulas $\Box .\overset{n}{\ldots}. \Box\varphi$ for each $n \geq 0$ and each axiom φ of L and *modus ponens* as the only inference rule.

Examples of (non-)weakly implicative logics – 3

- Global modal logics are weakly implicative (using the axiom $(\mathrm{K}_\Box)$ and the rule of necessitation).
- Local modal logics are not weakly implicative. Indeed, let L be any such logic and assume that $\overline{1} \to \varphi, \varphi \to \overline{1} \vdash_{\mathrm{L}} \Box\overline{1} \to \Box\varphi$. Since L expands CL, we know that

  $$\vdash_{\mathrm{L}} \varphi \to \overline{1} \qquad \varphi \vdash_{\mathrm{L}} \overline{1} \to \varphi \qquad \vdash_{\mathrm{L}} \overline{1}.$$

  Thus also $\vdash_{\mathrm{L}} \Box\overline{1}$ and so $\varphi \vdash_{\mathrm{L}} \Box\varphi$, i.e., L is equal to its global variant, which is known not be the case.

Congruence Property – 1

Conventions

Unless said otherwise, L is a weakly implicative in a language $\mathcal{L}$ with an implication $\to$. We write:

- $\varphi \leftrightarrow \psi$ instead of $\{\varphi \to \psi, \psi \to \varphi\}$
- $\Gamma \vdash \Delta$ whenever $\Gamma \vdash \chi$ for each $\chi \in \Delta$
- $\Gamma \dashv\vdash \Delta$ whenever $\Gamma \vdash \Delta$ and $\Delta \vdash \Gamma$.

Theorem

Let φ, ψ, χ be formulas. Then:

- $\vdash_{\mathrm{L}} \varphi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \psi \leftrightarrow \varphi$
- $\varphi \leftrightarrow \delta, \delta \leftrightarrow \psi \vdash_{\mathrm{L}} \varphi \leftrightarrow \psi$
- $\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} \chi \leftrightarrow \hat{\chi}$, *where $\hat{\chi}$ is obtained from χ by replacing some occurrences of φ in χ by ψ.*

Congruence Property – 2

Corollary

Let $\to'$ *be a connective satisfying* (R), (MP), (T), (sCng). *Then*

$$\varphi \leftrightarrow \psi \dashv\vdash_{\mathrm{L}} \varphi \leftrightarrow' \psi.$$

Leibniz congruence – 1

Let us fix a weakly implicative logic L.

Definition

Let $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ be an L-matrix. We define:

- the matrix preorder $\leq_{\mathbf{A}}$ of $\mathbf{A}$ as

$$a \leq_{\mathbf{A}} b \qquad \text{iff} \qquad a \to^{\boldsymbol{A}} b \in F$$

- the Leibniz congruence $\Omega_{\boldsymbol{A}}(F)$ of $\mathbf{A}$ as

$$\langle a, b \rangle \in \Omega_{\boldsymbol{A}}(F) \qquad \text{iff} \qquad a \leq_{\mathbf{A}} b \ \text{ and } \ b \leq_{\mathbf{A}} a.$$

A congruence θ of $\boldsymbol{A}$ is logical in a matrix $\langle \boldsymbol{A}, F \rangle$ if for each $a, b \in A$ if $a \in F$ and $\langle a, b \rangle \in \theta$, then $b \in F$.

Leibniz congruence – 2

Theorem

Let $\mathbf{A} = \langle A, F \rangle$ *be an* L*-matrix. Then:*

1. $\leq_{\mathbf{A}}$ *is a preorder.*
2. $\Omega_{\boldsymbol{A}}(F)$ *is the largest logical congruence of* $\langle \boldsymbol{A}, F \rangle$*.*
3. $\langle a, b \rangle \in \Omega_{\boldsymbol{A}}(F)$ *iff for each* $\chi \in Fm_{\mathcal{L}}$ *and each* $\boldsymbol{A}$*-evaluation* e*:*

$$e[p : a](\chi) \in F \qquad \textit{iff} \qquad e[p : b](\chi) \in F.$$

Leibniz congruence – 2

Theorem

Let $\mathbf{A} = \langle A, F\rangle$ *be an* L*-matrix. Then:*

1. $\leq_{\mathbf{A}}$ *is a preorder.*
2. $\Omega_{\mathbf{A}}(F)$ *is the largest logical congruence of* $\langle A, F\rangle$*.*
3. $\langle a, b\rangle \in \Omega_{\mathbf{A}}(F)$ *iff for each* $\chi \in Fm_{\mathcal{L}}$ *and each* $\mathbf{A}$*-evaluation* e*:*

$$e[p : a](\chi) \in F \qquad \textit{iff} \qquad e[p : b](\chi) \in F.$$

Proof.

1. Take A-evaluation e such that $e(p) = a$, $e(q) = b$, and $e(r) = c$. Recall that in L we have: $\vdash_{\mathrm{L}} p \to p$ and $p \to q, q \to r \vdash_{\mathrm{L}} p \to r$. As $\mathbf{A} = \mathbf{MOD}(\mathrm{L})$ we have: $e(p \to p) \in F$, i.e., $a \leq_{\mathbf{A}} a$ and
if $e(p \to q), e(q \to r) \in F$, then $e(p \to r) \in F$ i.e., if $a \leq_{\mathbf{A}} b$ and $b \leq_{\mathbf{A}} c$, then $a \leq_{\mathbf{A}} c$.

Leibniz congruence – 2

Theorem

Let $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ *be an* L*-matrix. Then:*

1. $\leq_{\mathbf{A}}$ *is a preorder.*
2. $\Omega_{\boldsymbol{A}}(F)$ *is the largest logical congruence of* $\langle \boldsymbol{A}, F\rangle$*.*
3. $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$ *iff for each* $\chi \in Fm_{\mathcal{L}}$ *and each* $\boldsymbol{A}$*-evaluation* e*:*

$$e[p:a](\chi) \in F \qquad \textit{iff} \qquad e[p:b](\chi) \in F.$$

Proof.

2. $\Omega_{\boldsymbol{A}}(F)$ is obviously an equivalence relation. It is a congruence due to (sCng) and logical due to (MP).

Take a logical congruence θ and $\langle a, b\rangle \in \theta$. Since $\langle a, a\rangle \in \theta$, we have $\langle a \to^{\boldsymbol{A}} a, a \to^{\boldsymbol{A}} b\rangle \in \theta$. As $a \to^{\boldsymbol{A}} a \in F$ and θ is logical we get $a \to^{\boldsymbol{A}} b \in F$, i.e., $a \leq_{\mathbf{A}} b$. The proof of $b \leq_{\mathbf{A}} a$ is analogous.

Leibniz congruence – 2

Theorem

Let $\mathbf{A} = \langle A, F\rangle$ *be an* L*-matrix. Then:*

1. $\leq_{\mathbf{A}}$ *is a preorder.*
2. $\Omega_{A}(F)$ *is the largest logical congruence of* $\langle A, F\rangle$*.*
3. $\langle a, b\rangle \in \Omega_{A}(F)$ *iff for each* $\chi \in Fm_{\mathcal{L}}$ *and each* A*-evaluation* e*:*

$$e[p : a](\chi) \in F \qquad \text{iff} \qquad e[p : b](\chi) \in F.$$

Proof.

3. One direction is a corollary of the congruence property and (MP). The converse one: set $\chi = p \to q$ and $e(q) = b$. Then, $a \to^{A} b \in F$ iff $b \to^{A} b \in F$, thus $a \leq_{\mathbf{A}} b$. The proof of $b \leq_{\mathbf{A}} a$ is analogous (using $e(q) = a$). □

Algebraic counterpart

Definition

An L-matrix $\mathbf{A} = \langle \boldsymbol{A}, F \rangle$ is reduced, $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ in symbols, if $\Omega_{\boldsymbol{A}}(F)$ is the identity relation $\mathrm{Id}_{\boldsymbol{A}}$.

An algebra $\boldsymbol{A}$ is L-algebra, $\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{L})$ in symbols, if there is a set $F \subseteq A$ such that $\langle \boldsymbol{A}, F \rangle \in \mathbf{MOD}^*(\mathrm{L})$.

Note that $\Omega_{\boldsymbol{A}}(A) = A^2$. Thus from $\mathcal{F}i_{\mathrm{Inc}}(\boldsymbol{A}) = \{A\}$ we obtain:

$$\boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{Inc}) \qquad \text{iff} \qquad A \text{ is a singleton}$$

Examples: classical logic CL and logic BCI

Exercise 4

Classical logic: prove that for any Boolean algebra $\boldsymbol{A}$:

$$\Omega_{\boldsymbol{A}}(\{1\}) = \mathrm{Id}_{A} \qquad \text{i.e., } \boldsymbol{A} \in \mathbf{ALG}^*(\mathrm{CL}).$$

On the other hand, show that:

$$\Omega_{\boldsymbol{4}}(\{a,1\}) = \mathrm{Id}_{A} \cup \{\langle 1,a\rangle, \langle 0,\neg a\rangle\} \text{ i.e. } \langle \boldsymbol{4}, \{a,1\}\rangle \notin \mathbf{MOD}^*(\mathrm{CL}).$$

BCI: recall the algebra $\boldsymbol{M}$ defined via:

$\to^{\boldsymbol{M}}$	$\top$	t	f	$\bot$
$\top$	$\top$	$\bot$	$\bot$	$\bot$
t	$\top$	t	f	$\bot$
f	$\top$	$\bot$	t	$\bot$
$\bot$	$\top$	$\top$	$\top$	$\top$

Show that:

$$\Omega_{\boldsymbol{M}}(\{t,\top\}) = \Omega_{\boldsymbol{M}}(\{t,f,\top\}) = \mathrm{Id}_{M} \qquad \text{i.e. } \quad \boldsymbol{M} \in \mathbf{ALG}^*(\mathrm{BCI}).$$

Factorizing matrices – 1

Let us take $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$. We write:

- $\boldsymbol{A}^*$ for $\boldsymbol{A}/\Omega_{\boldsymbol{A}}(F)$
- $[\cdot]_F$ for the canonical epimorphism of $\boldsymbol{A}$ onto $\boldsymbol{A}^*$ defined as:

$$[a]_F = \{b \in A \mid \langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)\}$$

- $\mathbf{A}^*$ for $\langle \boldsymbol{A}^*, [F]_F\rangle$.

Lemma

Let $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$ *and* $a, b \in A$. *Then:*

1. $a \in F$ *iff* $[a]_F \in [F]_F$.
2. $\mathbf{A}^* \in \mathbf{MOD}(\mathrm{L})$.
3. $[a]_F \leq_{\mathbf{A}^*} [b]_F$ *iff* $a \to^{\boldsymbol{A}} b \in F$.
4. $\mathbf{A}^* \in \mathbf{MOD}^*(\mathrm{L})$.

Factorizing matrices – 2

Proof.

1. One direction is trivial. Conversely: $[a]_F \in [F]_F$ implies that $[a]_F = [b]_F$ for some $b \in F$; thus $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$ and, since $\Omega_{\boldsymbol{A}}(F)$ is a logical congruence, we obtain $a \in F$.
2. Recall that the second claim of Lemma 1.12 says that for a surjective $g\colon \boldsymbol{A} \to \boldsymbol{B}$ and $F \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$ we get $g[F] \in \mathcal{F}i_{\mathrm{L}}(\boldsymbol{B})$, whenever $g(x) \in g[F]$ implies $x \in F$.
3. $[a]_F \leq_{\mathbf{A}^*} [b]_F$ iff $[a]_F \to^{\boldsymbol{A}^*} [b]_F \in [F]_F$ iff $[a \to^{\boldsymbol{A}} b]_F \in [F]_F$ iff $a \to^{\boldsymbol{A}} b \in F$.
4. Assume that $\langle [a]_F, [b]_F\rangle \in \Omega_{\boldsymbol{A}^*}([F]_F)$, i.e., $[a]_F \leq_{\mathbf{A}^*} [b]_F$ and $[b]_F \leq_{\mathbf{A}^*} [a]_F$. Therefore $a \to^{\boldsymbol{A}} b \in F$ and $b \to^{\boldsymbol{A}} a \in F$, i.e., $\langle a, b\rangle \in \Omega_{\boldsymbol{A}}(F)$. Thus $[a]_F = [b]_F$. □

Lindenbaum–Tarski matrix

Let L be a weakly implicative logic in $\mathcal{L}$ and $T \in \mathit{Th}(\mathrm{L})$. For every formula φ, we define the set

$$[\varphi]_T = \{\psi \in \mathit{Fm}_{\mathcal{L}} \mid \varphi \leftrightarrow \psi \subseteq T\}.$$

The Lindenbaum–Tarski matrix with respect to L and T, $\mathbf{LindT}_T$, has the filter $\{[\varphi]_T \mid \varphi \in T\}$ and algebraic reduct with the domain $\{[\varphi]_T \mid \varphi \in \mathit{Fm}_{\mathcal{L}}\}$ and operations:

$$c^{\mathbf{LindT}_T}([\varphi_1]_T, \ldots, [\varphi_n]_T) = [c(\varphi_1, \ldots, \varphi_n)]_T$$

Clearly, for every $T \in \mathrm{Th}(\mathrm{L})$ we have:

$$\mathbf{LindT}_T = \langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle^*.$$

The second completeness theorem

Theorem

Let L *be a weakly implicative logic. Then for any set* Γ *of formulas and any formula* φ *the following holds:*

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi.$$

The second completeness theorem

Theorem

Let L *be a weakly implicative logic. Then for any set* Γ *of formulas and any formula* φ *the following holds:*

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi.$$

Proof.

Using just the soundness part of the first completeness theorem it remains to prove:

$$\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi \quad \text{implies} \quad \Gamma \vdash_{\mathrm{L}} \varphi.$$

Take the Lindenbaum–Tarski matrix $\mathbf{LindT}_{\mathrm{Th_L}(\Gamma)} = \langle \boldsymbol{Fm}_{\mathcal{L}}, \mathrm{Th_L}(\Gamma)\rangle^*$ and evaluation $e(\psi) = [\psi]_{\mathrm{Th_L}(\Gamma)}$. As clearly $e[\Gamma] \subseteq e[\mathrm{Th_L}(\Gamma)] = [\mathrm{Th_L}(\Gamma)]_{\mathrm{Th_L}(\Gamma)}$, then, as $\mathbf{LindT}_{\mathrm{Th_L}(\Gamma)}$ is an L-model, we have:
$e(\varphi) = [\varphi]_{\mathrm{Th_L}(\Gamma)} \in [\mathrm{Th_L}(\Gamma)]_{\mathrm{Th_L}(\Gamma)}$, and so $\varphi \in \mathrm{Th_L}(\Gamma)$ i.e., $\Gamma \vdash_{\mathrm{L}} \varphi$. □

Completeness theorem for classical logic

- Suppose that $T \in \mathrm{Th}(\mathrm{CL})$ and $\varphi \notin T$ ($T \nvdash_{\mathrm{CL}} \varphi$). We want to show that $T \not\models \varphi$ in some meaningful semantics.
- $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}, T\rangle} \varphi$. 1st completeness theorem
- $\langle \alpha, \beta \rangle \in \Omega(T)$ iff $\alpha \leftrightarrow \beta \in T$ (congruence relation on $\boldsymbol{Fm}_{\mathcal{L}}$ compatible with T: if $\alpha \in T$ and $\langle \alpha, \beta \rangle \in \Omega(T)$, then $\beta \in T$).
- Lindenbaum–Tarski algebra: $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T)$ is a Boolean algebra and $T \not\models_{\langle \boldsymbol{Fm}_{\mathcal{L}}/\Omega(T), T/\Omega(T)\rangle} \varphi$.

  2nd completeness theorem
- Lindenbaum Lemma: If $\varphi \notin T$, then there is a maximal consistent $T' \in \mathrm{Th}(\mathrm{CL})$ such that $T \subseteq T'$ and $\varphi \notin T'$.
- $\boldsymbol{Fm}_{\mathcal{L}}/\Omega(T') \cong \boldsymbol{2}$ (subdirectly irreducible Boolean algebra) and $T \not\models_{\langle \boldsymbol{2}, \{1\}\rangle} \varphi$. 3rd completeness theorem

Closure systems and closure operators – 1

Closure system over a set A: a collection of subsets $\mathcal{C} \subseteq \mathcal{P}(A)$ closed under arbitrary intersections and such that $A \in \mathcal{C}$. The elements of $\mathcal{C}$ are called closed sets.

Closure operator over a set A: a mapping $C\colon \mathcal{P}(A) \to \mathcal{P}(A)$ such that for every $X, Y \subseteq A$:

1. $X \subseteq C(X)$,
2. $C(X) = C(C(X))$, and
3. if $X \subseteq Y$, then $C(X) \subseteq C(Y)$.

Exercise 5

If C is a closure operator, $\{X \subseteq A \mid C(X) = X\}$ is a closure system.

If $\mathcal{C}$ is closure system, $C(X) = \bigcap\{Y \in \mathcal{C} \mid X \subseteq Y\}$ is a closure operator.

Closure systems and closure operators – 2

A base of a closure system $\mathcal{C}$ over A is any $\mathcal{B} \subseteq \mathcal{C}$ satisfying one of the following equivalent conditions:

1. $\mathcal{C}$ is the coarsest closure system containing $\mathcal{B}$.
2. For every $T \in \mathcal{C}$, there is a $\mathcal{D} \subseteq \mathcal{B}$ such that $T = \bigcap \mathcal{D}$.
3. For every $T \in \mathcal{C}$, $T = \bigcap\{B \in \mathcal{B} \mid T \subseteq B\}$.
4. For every $Y \in \mathcal{C}$ and $a \in A \setminus Y$ there is $Z \in \mathcal{B}$ such that $Y \subseteq Z$ and $a \notin Z$.

Exercise 6

Show that the four definitions are equivalent.

An element X of a closure system $\mathcal{C}$ over A is called (finitely) $\cap$-irreducible if for each (finite non-empty) set $\mathcal{Y} \subseteq \mathcal{C}$ such that $X = \bigcap_{Y \in \mathcal{Y}} Y$, there is $Y \in \mathcal{Y}$ such that $X = Y$.

Abstract Lindenbaum Lemma

An element X of a closure system $\mathcal{C}$ over A is called maximal w.r.t. an element a if it is a maximal element of the set $\{Y \in \mathcal{C} \mid a \notin Y\}$ w.r.t. the order given by inclusion.

Proposition

Let $\mathcal{C}$ be a closure system over a set A and $T \in \mathcal{C}$. Then, T is maximal w.r.t. an element if, and only if, T is $\cap$-irreducible.

A closure operator C is finitary if for every $X \subseteq A$,
$C(X) = \bigcup\{C(Y) \mid Y \subseteq X$ and Y is finite$\}$.

Lemma

Let C be a finitary closure operator and $\mathcal{C}$ its corresponding closure system. If $T \in \mathcal{C}$ and $a \notin T$, then there is $T' \in \mathcal{C}$ such that $T \subseteq T'$ and T' is maximal with respect to a. $\cap$-irreducible closed sets form a base.

Operations on matrices

$\langle \boldsymbol{A}, F\rangle$: first-order structure in the equality-free predicate language with function symbols from $\mathcal{L}$ and a unique unary predicate symbol interpreted by F.

Submatrix: $\langle \boldsymbol{A}, F\rangle \subseteq \langle \boldsymbol{B}, G\rangle$ if $\boldsymbol{A} \subseteq \boldsymbol{B}$ and $F = A \cap G$. Operator: $\mathbf{S}$.
Homomorphic image: A homomorphism from $\langle \boldsymbol{A}, F\rangle$ to $\langle \boldsymbol{B}, G\rangle$ is a homomorphism of algebras $h\colon \boldsymbol{A} \to \boldsymbol{B}$ such that $h[F] \subseteq G$.
Direct product: $\langle \boldsymbol{A}, F\rangle = \prod_{i\in I}\{\langle \boldsymbol{A}_i, F_i\rangle \mid i \in I\}$ if $\boldsymbol{A} = \prod_{i\in I} \boldsymbol{A}_i$, $f^{\boldsymbol{A}}(a_1,\dots,a_n)(i) = f^{\boldsymbol{A}_i}(a_1(i),\dots,a_n(i))$. $F = \prod_{i\in I} F_i$. $\pi_j : \boldsymbol{A} \twoheadrightarrow \boldsymbol{A}_j$.
Operator: $\mathbf{P}$.

Exercise 7

Let L be a weakly implicative logic. Then:

1. $\mathbf{SP}(\mathbf{MOD}(\mathrm{L})) \subseteq \mathbf{MOD}(\mathrm{L})$.
2. $\mathbf{SP}(\mathbf{MOD}^*(\mathrm{L})) \subseteq \mathbf{MOD}^*(\mathrm{L})$.

Subdirect products and subdirect irreducibility

$\mathbf{A}$ is representable as a subdirect product of $\{\mathbf{A}_i \mid i \in I\}$ if there is an embedding α from $\mathbf{A}$ into $\prod_{i\in I} \mathbf{A}_i$ s.t. for every $i \in I$, $\pi_i \circ \alpha$ is a surjective homomorphism.

Operator $\mathbf{P}_{\mathrm{SD}}(\mathbb{K})$.

$\mathbf{A} \in \mathbb{K}$ is (finitely) subdirectly irreducible relative to $\mathbb{K}$ if for every (finite non-empty) subdirect representation α of $\mathbf{A}$ with a family $\{\mathbf{A}_i \mid i \in I\} \subseteq \mathbb{K}$ there is $i \in I$ such that $\pi_i \circ \alpha$ is an isomorphism.

The class of all (finitely) subdirectly irreducible matrices relative to $\mathbb{K}$ is denoted as $\mathbb{K}_{\mathrm{R(F)SI}}$.

$\mathbb{K}_{\mathrm{RSI}} \subseteq \mathbb{K}_{\mathrm{RFSI}}$.

Characterization of RSI and RFSI reduced models

Theorem

Given a weakly implicative logic L *and* $\mathbf{A} = \langle \boldsymbol{A}, F\rangle \in \mathbf{MOD}^*(\mathrm{L})$, *we have:*

1. $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ *iff* F *is* $\cap$*-irreducible in* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.
2. $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ *iff* F *is finitely* $\cap$*-irreducible in* $\mathcal{F}i_{\mathrm{L}}(\boldsymbol{A})$.

Subdirect representation

Theorem

If L *is a finitary weakly implicative logic, then*

$$\mathbf{MOD}^*(\mathrm{L}) = \mathbf{P}_{\mathrm{SD}}(\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}),$$

in particular every matrix in $\mathbf{MOD}^*(\mathrm{L})$ *is representable as a subdirect product of matrices in* $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$*.*

The third completeness theorem

Theorem

Let L *be a finitary weakly implicative logic. Then for any set* Γ *of formulas and any formula* φ *the following holds:*

$$\Gamma \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}} \varphi.$$

Outline

Non-associative residuated lattices [Galatos–Ono. APAL 2010]

A pointed residuated lattice-ordered groupoid with unit $\boldsymbol{A}$ is algebra of a type $\mathcal{L}_{\mathrm{SL}} = \{\&, \backslash, /, \wedge, \vee, \overline{0}, \overline{1}\}$:

- $\langle A, \wedge, \vee \rangle$ is a lattice
- $\langle A, \&, \overline{1} \rangle$ is a groupoid with unit $\overline{1}$
- for each $x, y, z \in A$:

$$x \,\&\, y \leq z \quad \text{IFF} \quad x \leq z \,/\, y \quad \text{IFF} \quad y \leq x \,\backslash\, z$$

For simplicity we will speak about SL-algebras

SL-algebras form a variety, we will denote it as $\mathbb{SL}$.

Classes of residuated structures

Any quasivariety of $\mathbb{SL}$-algebras, possibly with additional operators, will be called a class of residuated structures.

Classes of residuated structures

Any quasivariety of SL-algebras, possibly with additional operators, will be called a class of residuated structures.

- Subvarieties of $\mathbb{SL}$, where $\&$ is associative, commutative, idempotent, divisible, etc.
- Integral SL-algebras: those where $\overline{1}$ is a top element of A
- Semilinear classes (those generated by their linearly ordered members)
- G-algebras (associative, commutative, integral, semilinear SL-algebras where $x \,\&\, y = x \wedge y$)
- MV-algebras (associative, commutative, integral, divisible, semilinear SL-algebras where $(x \to \overline{0}) \to \overline{0} = x$)
- Boolean algebras (idempotent MV-algebras)

Plus any of these with additional operators ...

The logic of $\mathbb{SL}$-algebras

The relation $\vdash_{\mathrm{SL}}$ defined as:

$$\Gamma \vdash_{\mathrm{SL}} \varphi \quad \text{iff} \quad \{\psi \wedge \overline{1} \approx \overline{1} \mid \psi \in \Gamma\} \models_{\mathbb{SL}} \varphi \wedge \overline{1} \approx \overline{1}$$

is a logic.

The logic of $\mathbb{SL}$-algebras

The relation $\vdash_{\mathrm{SL}}$ defined as:

$$\Gamma \vdash_{\mathrm{SL}} \varphi \quad \text{iff} \quad \{\psi \geq \overline{1} \mid \psi \in \Gamma\} \models_{\mathbb{SL}} \varphi \geq \overline{1}$$

is a logic.

Axiomatization $\mathcal{SL}$ for SL [Galatos–Ono. APAL, 2010]

Axioms:

$$\begin{array}{lll}
\varphi \wedge \psi \setminus \varphi & \varphi \wedge \psi \setminus \psi & (\chi \setminus \varphi) \wedge (\chi \setminus \psi) \setminus (\chi \setminus \varphi \wedge \psi) \\
\varphi \setminus \varphi \vee \psi & \psi \setminus \varphi \vee \psi & (\varphi \setminus \chi) \wedge (\psi \setminus \chi) \setminus (\varphi \vee \psi \setminus \chi) \\
\varphi \setminus ((\psi \,/\, \varphi) \setminus \psi) & \psi \setminus (\varphi \setminus \varphi \,\&\, \psi) & (\chi \,/\, \varphi) \wedge (\chi \,/\, \psi) \setminus (\chi \,/\, \varphi \vee \psi) \\
\overline{1} & \overline{1} \setminus (\varphi \setminus \varphi) & \varphi \setminus (\overline{1} \setminus \varphi)
\end{array}$$

Rules:

$$\begin{array}{ll}
\{\varphi, \varphi \setminus \psi\} \rhd \psi & \{\varphi\} \rhd (\varphi \setminus \psi) \setminus \psi \\
\{\varphi \setminus (\psi \setminus \chi)\} \rhd \psi \setminus (\chi \,/\, \varphi) & \{\psi \,/\, \varphi\} \rhd \varphi \setminus \psi \\
\{\varphi \setminus \psi\} \rhd (\psi \setminus \chi) \setminus (\varphi \setminus \chi) & \{\psi \setminus \chi\} \rhd (\varphi \setminus \psi) \setminus (\varphi \setminus \chi) \\
\{\varphi, \psi\} \rhd \varphi \wedge \psi & \{\psi \setminus (\varphi \setminus \chi)\} \rhd \varphi \,\&\, \psi \setminus \chi
\end{array}$$

A formal definition of substructural logics

We write $\varphi \to \psi$ instead of $\varphi \backslash \psi$

$\varphi \leftrightarrow \psi$ instead of $(\varphi \to \psi) \wedge (\psi \to \varphi)$

Definition

A finitary logic L in a language $\mathcal{L}$ is a substructural logic if

- $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{SL}}$
- If $T \vdash_{\mathrm{SL}} \varphi$, then $T \vdash_{\mathrm{L}} \varphi$
- for each n, $i < n$, and each n-ary connective $c \in \mathcal{L} \setminus \mathcal{L}_{\mathrm{SL}}$ holds:

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}} c(\chi_1, \dots \chi_i, \varphi, \dots, \chi_n) \leftrightarrow c(\chi_1, \dots \chi_i, \psi, \dots, \chi_n)$$

The last condition can be proven for all connectives of $\mathcal{L}_{\mathrm{SL}}$. Hence, all substructural logics are weakly implicative.

From substructural logics to classes of residuated structures

Theorem

Let L *be a substructural logic. We say that an* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *is an* L*-algebra, whenever*

1. *its* $\mathcal{L}_{\mathrm{SL}}$*-reduct is an* SL*-algebra and*
2. $T \vdash_{\mathrm{L}} \varphi$ *implies that* $\{\psi \geq \overline{1} \mid \psi \in T\} \models_{\boldsymbol{A}} \varphi \geq \overline{1}$

The class of all L*-algebras, denoted as* $\mathbb{Q}_{\mathrm{L}}$*, is a class of residuated structures and*

$$T \vdash_{\mathrm{L}} \varphi \quad \textit{iff} \quad \{\psi \geq \overline{1} \mid \psi \in T\} \models_{\mathbb{Q}_{\mathrm{L}}} \varphi \geq \overline{1}$$

From substructural logics to classes of residuated structures and back

Theorem

Let $\mathbb{Q}$ be a class of residuated structures of type $\mathcal{L} \supseteq \mathcal{L}_{\mathrm{SL}}$. Then the relation $\mathrm{L}_{\mathbb{Q}}$ defined as:

$$T \vdash_{\mathrm{L}_{\mathbb{Q}}} \varphi \quad \textit{iff} \quad \{\psi \geq \overline{1} \mid \psi \in T\} \models_{\mathbb{Q}} \varphi \geq \overline{1}$$

is a substructural logic. Moreover:

$$E \models_{\mathbb{Q}} \alpha \approx \beta \quad \textit{iff} \quad \{\varphi \leftrightarrow \psi \mid \varphi \approx \psi \in E\} \vdash_{\mathrm{L}_{\mathbb{Q}}} \alpha \leftrightarrow \beta$$

It gets even better

Theorem

The operators $\mathbb{Q}_\star$ and $\mathrm{L}_\star$ are dual-lattice isomorphisms between the lattice of substructural logics in language $\mathcal{L}$ and the lattice of subquasivarieties of SL-algebras with operators $\mathcal{L} \setminus \mathcal{L}_{\mathrm{SL}}$.

Examples of substructural logics

- CL, IL, G, Ł, etc.
- expansions by additional connectives, e.g. (classical) modalities, exponentials in linear logic and Baaz's Delta in fuzzy logics

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Special axioms:

usual name	**s**	axioms
associativity	a	$(\varphi \& \psi) \& \chi \leftrightarrow \varphi \& (\psi \& \chi)$
exchange	e	$\varphi \& \psi \to \psi \& \varphi$
contraction	c	$\varphi \to \varphi \& \varphi$
weakening	w	$\varphi \& \psi \to \psi$ and $\overline{0} \to \varphi$

Logic given by these axioms; let $X \subseteq \{e, c, w\}$ we define logics

- SL_X axiomatized by adding axioms from X of those of SL
- FL_X axiomatized by adding associativity to SL_X

Proof by cases

For classical or intuitionistic logic we have:

$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \chi \qquad\qquad \Gamma, \psi \vdash_{\mathrm{L}} \chi}{\Gamma \cup \{\varphi \vee \psi\} \vdash_{\mathrm{L}} \chi}$$

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But in $\mathrm{FL_e}$ it would entail $\varphi \vee \psi \vdash_{\mathrm{FL_e}} (\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1})$, i.e.,

$$(\varphi \vee \psi) \wedge \overline{1} \approx \overline{1} \models_{\mathbb{Q}_{\mathrm{FL_e}}} (\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1}) \approx \overline{1}$$

which can be easily refuted

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On the other hand, we can show that:

$$\frac{\Gamma, \varphi \vdash_{\mathrm{FL_e}} \chi \qquad\qquad \Gamma, \psi \vdash_{\mathrm{FL_e}} \chi}{\Gamma \cup \{(\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1})\} \vdash_{\mathrm{FL_e}} \chi}$$

Generalized disjunctions

Let $\nabla(p, q, \vec{r})$ be a set of formulas. We write

$$\varphi \nabla \psi = \bigcup \{\nabla(\varphi, \psi, \vec{\alpha}) \mid \vec{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{\leq\omega}\}.$$

Definition

∇ is a p-disjunction if:

(PD)	$\varphi \vdash_{\mathrm{L}} \varphi \nabla \psi$	and	$\psi \vdash_{\mathrm{L}} \varphi \nabla \psi$		
PCP	$\Gamma, \varphi \vdash_{\mathrm{L}} \chi$	and	$\Gamma, \psi \vdash_{\mathrm{L}} \chi$	implies	$\Gamma, \varphi \nabla \psi \vdash_{\mathrm{L}} \chi$

Definition

A logic L is a p-disjunctional if it has a p-disjunction.

We drop the prefix 'p-' if there are no parameters $\vec{r}$ in ∇

Separating examples

Example

- $\vee$ is a disjunction in $\mathrm{FL_{ew}}$
- $\vee$ is not a disjunction in $\mathrm{FL_{e}}$,

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- No finite set of formulas is a disjunction in K

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  but the set $\{(p \rightarrow q) \rightarrow q, (q \rightarrow p) \rightarrow p\}$ is
- No finite set of formulas is a disjunction in K
  but the set $\{\Box^n p \vee \Box^m q \mid n, m \geq 0\}$ is
- No set of formulas in two variables is a disjunction in $\mathrm{IL}_{\rightarrow}$

Separating examples

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- No single formula is a disjunction in $\mathrm{G}_{\to}$
  but the set $\{(p \to q) \to q, (q \to p) \to p\}$ is
- No finite set of formulas is a disjunction in K
  but the set $\{\Box^n p \vee \Box^m q \mid n, m \geq 0\}$ is
- No set of formulas in two variables is a disjunction in $\mathrm{IL}_{\to}$
  but the formula $(p \to r) \to ((q \to r) \to r)$ is a p-disjunction.

Filters in p-disjunctional logics

Theorem

Let L *be a logic with a p-disjunction* ∇*. Then for each* $\mathcal{L}$*-algebra* $\boldsymbol{A}$ *and each* $X, Y \cup \{x, y\} \subseteq A$*:*

$$\mathrm{Fi}(X, x) \cap \mathrm{Fi}(X, y) = \mathrm{Fi}(X, x \nabla^{\boldsymbol{A}} y)$$

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$$\mathrm{Fi}(X) \cap \mathrm{Fi}(Y) = \mathrm{Fi}(\{a \nabla^{\boldsymbol{A}} b \mid a \in X, b \in Y\}).$$

Theorem

Let L *be a substructural logic. TFAE:*

1. L *is p-disjunctional*
2. *The lattice of all* L*-filters on any* $\mathcal{L}$*-algebra is distributive*
3. $\mathbb{Q}_{\mathrm{L}}$ *is relative-congruence-distributive*

∇-prime filters

Definition

A filter F on $\boldsymbol{A}$ is ∇-prime if for every $a, b \in A$, $a \nabla^{\boldsymbol{A}} b \subseteq F$ implies $a \in F$ or $b \in F$.

Theorem

Let ∇ be a p-disjunction in L and $\boldsymbol{A}$ an L-algebra. Then,

$$\boldsymbol{A} \in (\mathbb{Q}_{\mathrm{L}})_{\mathrm{RFSI}} \text{ iff the filter } [\overline{1}\rangle \text{ is } \nabla\text{-prime.}$$

Semilinear logics

Let us denote by $\mathbb{Q}^{\ell}_{\mathrm{L}}$ the class of linearly ordered L-algebras.

Definition

A substructural logic L is called semilinear if

$$T \vdash_{\mathrm{L}} \varphi \quad \text{iff} \quad \{\psi \geq \overline{1} \mid \psi \in T\} \models_{\mathbb{Q}^{\ell}_{\mathrm{L}}} \varphi \geq \overline{1}$$

Characterizations of substructural semilinear logics

Theorem

Let L be a substructural logic. TFAE:

1. L *is semilinear*
2. $\mathbb{Q}_{\mathrm{L}} = \mathbf{Q}(\mathbb{Q}^{\ell}_{\mathrm{L}})$
3. $\mathbb{Q}^{\ell}_{\mathrm{L}} = (\mathbb{Q}_{\mathrm{L}})_{\mathrm{RFSI}}$
4. *Each* L*-algebra is a subdirect product of* L*-chains*
5. *Any* L*-filter in an* $\mathcal{L}$*-algebra is an intersection of linear ones*

   a filter F is *linear* if $x \to y \in F$ or $y \to x \in F$, for each x, y
6. *The following metarule holds:*

$$\frac{T, \varphi \to \psi \vdash_{\mathrm{L}} \chi \qquad T, \psi \to \varphi \vdash_{\mathrm{L}} \chi}{T \vdash_{\mathrm{L}} \chi}$$

Characterizations of substructural semilinear logics

Theorem

Let L *be a substructural logic and an axiomatic system* $\mathcal{AX}$. *TFAE:*

1. L *is semilinear,*
2. L *proves* $(\varphi \to \psi) \vee (\psi \to \varphi)$ *and enjoys the metarule:*

$$\frac{T, \varphi \vdash_{\mathrm{L}} \chi \qquad T, \psi \vdash_{\mathrm{L}} \chi}{T, \varphi \vee \psi \vdash_{\mathrm{L}} \chi}$$

3. L *proves* $(\varphi \to \psi) \vee (\psi \to \varphi)$ *and any* L*-filter in an* $\mathcal{L}$*-algebra is an intersection of* $\vee$*-prime ones,*
4. L *proves* $(\varphi \to \psi) \vee (\psi \to \varphi)$ *and for every rule* $T \rhd \varphi$ *in* $\mathcal{AX}$ *and propositional variable* p *not occurring in* T, φ *we have*

$$\{\psi \vee \chi \mid \psi \in T\} \vdash_{\mathrm{L}} \varphi \vee \chi$$

Wanna know more?

Forthcoming book:

P. Cintula, C.N. *Logic and Implication: An introduction to the general algebraic study of non-classical logics*, Trends in Logic, Springer.

Implication gives a nice bridge between logic and algebra . . .