Algebraic semantics for substructural logics
Constructions of residuated lattices.

Manuela Busaniche

Instituto de Matemática Aplicada del Litoral
Santa Fe, Argentina

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Dedicated to the memory of Roberto Cignoli

1erSLALM, Santiago de Chile, 1970

Language of LK

- The language of the propositional sequent calculus LK:

$$\mathcal{L} = \begin{cases} \text{proposicional variables} & x_1, x_2, \ldots x_n, \ldots, \\ \text{connectives} & \to, \vee, \wedge, \neg, \top \end{cases}$$

- The usual notion of formula.
- Sequent

$$\Gamma \Rightarrow \Delta$$

Sequent calculus LK

- Rules for connectives
- Initial sequent $\alpha \Rightarrow \alpha$
- The cut rule $\frac{\Gamma \Rightarrow \alpha \quad \alpha \Rightarrow \Pi}{\Gamma \Rightarrow \Pi}$

Sequent calculus LK

Structural rules

Weakening rule:

$$\frac{\Gamma \Rightarrow \Pi}{\Gamma, \alpha \Rightarrow \Pi} \qquad \frac{\Gamma \Rightarrow \Lambda}{\Gamma \Rightarrow \Lambda, \alpha}$$

Contraction rule:

$$\frac{\Gamma, \alpha, \alpha \Rightarrow \Pi}{\Gamma, \alpha \Rightarrow \Pi} \qquad \frac{\Gamma \Rightarrow \Lambda, \alpha, \alpha}{\Gamma \Rightarrow \Lambda, \alpha}$$

Exchange rule:

$$\frac{\Gamma, \alpha, \beta \Rightarrow \Pi}{\Gamma, \beta, \alpha \Rightarrow \Pi} \qquad \frac{\Gamma \Rightarrow \Lambda, \alpha, \beta}{\Gamma \Rightarrow \Lambda, \beta, \alpha}$$

Two important features of commas

- $\alpha \wedge \beta \Rightarrow \gamma$ is provable iff $\alpha, \beta \Rightarrow \gamma$
- A formula γ follows from α and β if and only if the implication $\alpha \rightarrow \gamma$ follows from β.

  Formally:

$$\alpha, \beta \Rightarrow \gamma \text{ iff } \beta \Rightarrow \alpha \rightarrow \gamma$$

Commas

If we replace the comma by a new logical connective (fusion) we have

$$\alpha * \beta \Rightarrow \gamma \text{ iff } \beta \Rightarrow \alpha \to \gamma$$

which in algebraic models can be expressed as

$$\alpha * \beta \leq \gamma \text{ iff } \beta \leq \alpha \to \gamma$$

Structural rules

Weakening rule:

$$\frac{\Gamma \Rightarrow \Pi}{\Gamma, \alpha \Rightarrow \Pi}$$

Contraction rule:

$$\frac{\Gamma, \alpha, \alpha \Rightarrow \Pi}{\Gamma, \alpha \Rightarrow \Pi}$$

Exchange rule:

$$\frac{\Gamma, \alpha, \beta \Rightarrow \Pi}{\Gamma, \beta, \alpha \Rightarrow \Pi}$$

Algebraic interpretation of the rules

- $\Rightarrow \dashrightarrow \leq$
- contraction rules $\dashrightarrow$ $x * x = x$
- exchange rule $\dashrightarrow$ $x * y = y * x$
- weakening rule $\dashrightarrow$ $x * y \leq x$

Implication

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}(\Rightarrow \to) \qquad \frac{\Gamma \Rightarrow \alpha \quad \beta \Rightarrow \Pi}{\Gamma, \alpha \to \beta \Rightarrow \Pi}(\to \ \Rightarrow)$$

The lack of the exchange rule will yield two implications:

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \beta / \alpha}(\Rightarrow \ /) \qquad \frac{\Gamma \Rightarrow \alpha \quad \beta \Rightarrow \Pi}{\Gamma, \beta / \alpha \Rightarrow \Pi}(/ \ \Rightarrow)$$

$$\frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \setminus \beta}(\Rightarrow \ \setminus) \qquad \frac{\Gamma \Rightarrow \alpha \quad \beta \Rightarrow \Pi}{\Gamma, \alpha \setminus \beta \Rightarrow \Pi}(\setminus \ \Rightarrow)$$

The residuation property in the presence of the exchange rule was:

$$x * y \leq z \quad \text{iff} \quad y \leq x \to z$$

without exchange $*, \backslash, /$ it becomes:

$$x * y \leq z \quad \text{iff} \quad x \leq z/y \quad \text{iff} \quad y \leq x \backslash z$$

Residuated lattices

A *residuated lattice* is an algebraic structure $\mathbf{A} = (A, *, \backslash, /, \wedge, \vee, e)$ of type $(2, 2, 2, 2, 2, 0)$ such that

- $(A, *, e)$ is a monoid
- $(A, \wedge, \vee)$ is a lattice
- $$x * y \leq z \text{ iff } x \leq z/y \text{ iff } y \leq x \backslash z \tag{1}$$
  for every $x, y, z \in A$, where $\leq$ is the partial order induced by the lattice structure.

Examples of residuated lattices

Lattice ordered groups

A lattice ordered group (l-group for short) is an algebra $\mathbf{G} = (G, \wedge, \vee, *, ^{-1}, e)$, such that $(G, \wedge, \vee)$ is a lattice, $(G, *, e)$ is a monoid, and for all $x \in G$ it satisfies

$$x * x^{-1} = e = x^{-1} * x$$

and $*$ preserves the order. But it can also be defined as a residuated lattice $(G, \wedge, \vee, *, \backslash, /, e)$ that satisfies that

$$(e/x)x = e = x(x \setminus e).$$

Examples of residuated lattices

The class of *l*-groups is a proper subvariety of the class of residuated lattices. The most famous examples are: $\mathbb{Z}$; $\mathbb{Q}$; $\mathbb{R}$ with the operation of addition and the usual order. These are all examples of commutative groups. An example of non-commutative *l*-group can be obtained by considering the set of automorphisms of a totally ordered set.

Examples of residuated lattices

Integral residuated lattices (weakening)

$\mathbf{A}$ is *integral* if $\forall x \in A$ we have $x \leq e$ $(e = \top)$.

Commutative residuated lattices (exchange)

$\mathbf{A}$ is *commutative* if $\forall x, y \in A$ we have $x * y = y * x$.
This being the case $/ = \backslash = \rightarrow$.

Brouwerian algebras (contraction and weakening)

$\mathbf{A}$ is *Brouwerian* if it is integral and $\forall x, y \in A$ we have $x \wedge y = x * y$.

Bounded commutative residuated lattices

A bounded commutative residuated lattice is an algebra

$$\mathbf{A} = \langle A, *, \to, \vee, \wedge, \top, \bot \rangle$$

such that $\langle A, *, \to, \vee, \wedge, \top \rangle$ is an integral and commutative residuated lattice, and $\bot$ is the smallest element of the lattice $\mathbf{L}(\mathbf{A})$.

In a bounded commutative residuated lattice one can define an extra operation

$$\neg x = x \to \bot$$

that has the properties of a negation.

Famous bounded residuated lattices and their corresponding logics

Boolean Algebras	$\longrightarrow$	Classical Propositional Logic
Heyting Algebras	$\longrightarrow$	Intuitionistic Logic
MV-algebras	$\longrightarrow$	Łukasiewicz Propositional Logic
BL-algebras	$\longrightarrow$	Basic Logic (Hájek's fuzzy logic)
MTL-algebras	$\longrightarrow$	Monoidal t-norm based logic
Nelson residuated lattices	$\longrightarrow$	Nelson constructive logic

Algebraic equivalents of logical notions

logics	$\longleftrightarrow$	varieties (or equational classes)
lattice of extensions	$\longleftrightarrow$	dual lattices of subvarieties
finite model property	$\longleftrightarrow$	generation by finite algebras
interpolation	$\longleftrightarrow$	amalgamation of algebras
Beth definability	$\longleftrightarrow$	surjectivity of epimorphisms
deduction theorem	$\longleftrightarrow$	equationally definable principal congruences

Challenge

- Challenge: Try to study and analyze different classes (subvarieties) of residuated lattices.
- Method to achieve our aim: Through constructions that use simpler or better known structures.

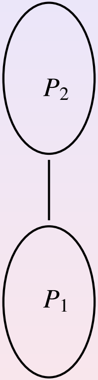
Constructions of residuated lattices

- Ordinal sums
- Poset products
- Rotations
- Twists products
- Representation by triples

Ordinal Sums and Poset Products

$I = \{1 < 2\}$

$$\bigoplus_{i \in I} P_i$$



Ordinal sums and poset products

- BL-algebras
- Hoops
- GBL-algebras
- MTL-algebras

BL-algebras

BL-algebras

They are bounded commutative residuated lattices that also satisfies

- Prelinearity $(x \to y) \vee (y \to x) = e$
- Divisibility $x * (x \to y) = y * (y \to x)$

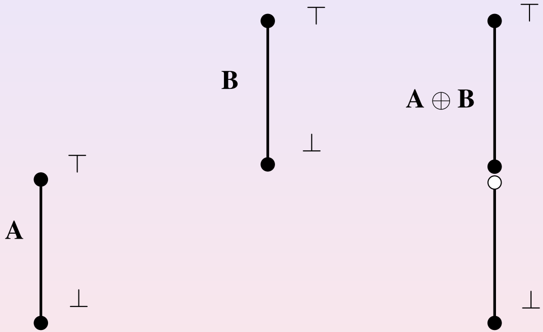
Particular cases of BL-algebras:

- Boolean algebras
- Prelinear Heyting algebras (Gödel algebras)
- MV-algebras
- Product algebras

- BL-algebras form a variety (equational class) of algebras
- Each BL-algebra is a subdirect product of totally ordered BL-algebras (BL-chains).

Ordinal sum and BL-algebras

Let **A** and **B** be two BL-chains:



x $\qquad$ x

$y = x * y$ $\qquad$ $y = x \to y$

Let $(I, \leq)$ be a totally ordered set. For each $i \in I$, let $\mathbf{A}_i = \langle A_i, *_i, \rightarrow_i, \top \rangle$ be a BL-chain such that for every $i \neq j$, $A_i \cap A_j = \{\top\}$.

Then we can define the ordinal sum as the BL-chain $\bigoplus_{i \in I} \mathbf{A}_i = \langle \cup_{i \in I} A_i, *, \rightarrow, \top \rangle$ where the operations $\cdot, \rightarrow$ are given by:

$$x * y = \begin{cases} x *_i y & \text{if } x, y \in A_i, \\ x & \text{if } x \in A_i \setminus \{\top\}, y \in A_j \text{ and } i < j, \\ y & \text{if } y \in A_i \setminus \{\top\}, x \in A_j \text{ and } i < j. \end{cases}$$

$$x \rightarrow y = \begin{cases} \top & \text{if } x \in A_i \setminus \{\top\},\ y \in A_j \text{ and } i < j, \\ x \rightarrow_i y & \text{if } x, y \in A_i, \\ y & \text{if } y \in A_i, x \in A_j \text{ and } i < j. \end{cases}$$

Decomposition theorem for BL-chains: Every totally ordered BL-algebra can be uniquely decomposed into an ordinal sum of structures known as Wajsberg hoops.

In every totally ordered Wajsberg hoop the product behaves as the product on a lattice ordered group.

Consequences of the decomposition:

subvarieties of BL-algebras	$\longleftrightarrow$	sublattice of logics
amalgamation of algebras	$\longleftrightarrow$	interpolation
complete algebras and completions	$\longleftrightarrow$	first ordered structures
description of fuzzy functions	$\longleftrightarrow$	characterization of formulas

Twist Structures and Rotations

Involutive residuated lattices

An *involution* on $\mathbf{L} \in CRL$ is a unary operation $\sim$ satisfying:

$$\sim\sim x = x \text{ and } x \to \sim y = y \to \sim x.$$

Twist-structures

- J. Kalman, *Lattices with involution*, Trans. Amer. Math. Soc. **87** (1958), 485–491.
- M. Kracht, *On extensions of intermediate logics by strong negation*, J. Philos. Log. **27** (1998), 49–73.

Given a lattice $\mathbf{L} = \langle L, \vee, \wedge \rangle$ the twist constructions are obtained by considering

$$\mathbf{L}^{twist} = \langle L \times L, \sqcup, \sqcap, \sim \rangle$$

with the operations $\sqcup, \sqcap$ given by

$$(a, b) \sqcup (c, d) = (a \vee c, b \wedge d) \tag{2}$$

$$(a, b) \sqcap (c, d) = (a \wedge c, b \vee d) \tag{3}$$

$$\sim (a, b) = (b, a) \tag{4}$$

The operation $\sim$ satisfies:

1. $\sim\sim x = x$
2. $\sim(x \sqcap y) = \sim x \sqcup \sim y$
3. $\sim(x \sqcup y) = \sim x \sqcap \sim y$

When the lattice $\mathbf{L}$ has some additional operations, the construction $\mathbf{L}^{twist}$ can also be endowed with some additional operations.

This construction has been used to represent some well-known algebras:

- Nelson algebras

  Fidel, Vakarelov,

  Sendlewski,Cignoli, . . .

- Involutive residuated lattices

  Tsinakis, Wille

  Galatos, Raftery, . . .

- N4-lattices

  Odintsov

- Bilattices

  Ginsberg, Fitting,

  Avron, Rivieccio, . . .

Let $\mathbf{L} = \langle L, \vee, \wedge, *, \to, e\rangle$ be an integral commutative residuated lattice. In $L \times L$ we can define the operations $\cdot, \to$ by

$$(a, b) * (c, d) = (a * c, (a \to d) \wedge (c \to b)) \tag{5}$$

$$(a, b) \to (c, d) = ((a \to c) \wedge (d \to b), a * d) \tag{6}$$

A particular case is then the algebra:

$$\mathbf{K}(\mathbf{L}) = \langle L \times L, \sqcup, \sqcap, *, \to, (e, e)\rangle$$

which is an involutive residuated lattice with

$$\sim (a, b) = (a, b) \to (e, e).$$

Definition

We call $\mathbf{K}(\mathbf{L})$ the *full twist-product* obtained from $\mathbf{L}$, and every subalgebra $\mathbf{A}$ of $\mathbf{K}(\mathbf{L})$ containing the set $\{(a, e) : a \in L\}$ is called *twist-product* obtained from $\mathbf{L}$.

For every integral commutative residuated lattice $\mathbf{L}$ the twist-products $\mathbf{K}(\mathbf{L})$ and their subalgebras form a variety that we call K-lattices (we denote by $\mathbb{K}$).

Theorem

Every K-lattice $\mathbf{A}$ *is isomorphic to a twist-product obtained from its negative cone.*

Categories

The application

$$\mathbf{L} \mapsto \mathbf{K}(\mathbf{L})$$

from

$$\mathbb{ICRL} \to \text{K-lattices}$$

defines a functor and the application

$$\mathbf{A} \mapsto \mathbf{A}^{-}$$

from

$$\text{K-lattices} \to \mathbb{ICRL}$$

is also a functor which is right adjoint to the first.

Categories

If we denote by $\mathcal{T}$ the full subcategory of K-lattices whose objects are the total K-lattices, i.e., $\mathbf{A} \cong \mathbf{K}(\mathbf{A}^-)$ then

Theorem

The categories of integral commutative residuated lattices and $\mathcal{T}$ are equivalent categories.

Applications

congruences of $\mathbf{K}(\mathbf{L})$	$\longleftrightarrow$	congruences of $\mathbf{L}$
equations on $\mathbf{K}(\mathbf{L})$	$\longleftrightarrow$	equations on $\mathbf{L}$
categorical constructions in $\mathbf{K}(\mathbf{L})$	$\longleftrightarrow$	categorical constructions in $\mathbf{L}$

Representation by triples

Representation by triples: Historical background

- Chen, C. C. and Grätzer, G., Stone Lattices. I: Construction Theorems, Canad. J. Math. *21* (1969).
- Katriňák, T., *A new proof of the construction theorem for Stone algebras, Proc. Amer. Math. Soc.*, **40** (1973).
- Maddana Swamy, U. and Rama Rao, V. V., *Triple and sheaf representations of Stone lattices*, Algebra Universalis **5** (1975).
- Montagna, F. and Ugolini, S., *A categorical equivalence for product algebras*, Studia Logica **103** (2015).
- Aguzzoli, S., Flaminio, T. and Ugolini, S, *Equivalences between the subcategories of MTL-algebras via boolean algebras and prelinear semihoops*, Journal of Logic and Computation, (2017).
- Busaniche, M., Cignoli, R. and Marcos, M., *A categorial equivalence for Stonean residuated lattices*, Studia Logica (2018).
- Busaniche, M., Marcos, M. and Ugolini, S., *Representation by triples of algebras with an MV-retract*, 22. Fuzzy Sets and Systems 369 (2019).

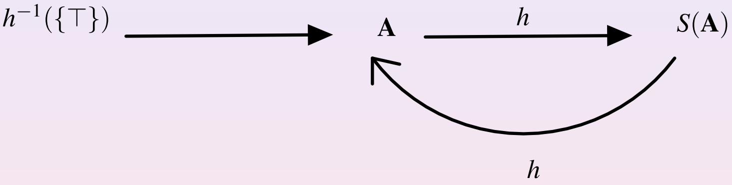
The retraction

Given a residuated lattice $\mathbf{A}$ a retraction is a homomorphism

$$h : A \twoheadrightarrow S(A)$$

onto a subalgebra $S(\mathbf{A})$ of $\mathbf{A}$ such that $h(h(a)) = h(a)$ for each $a \in A$.

If we have a class of residuated lattices with a retraction onto a subalgebra, we have the following situation:


$h^{-1}(\{\top\})$
$\mathbf{A}$
h
$S(\mathbf{A})$
h

- When can we use $S(\mathbf{A})$ and $h^{-1}(\{\top\})$ to recover $\mathbf{A}$?
- When is it worth using them?
- Are they all the information we need to characterize $\mathbf{A}$?

Stonean residuated lattices

Stonean residuated lattices is the greatest subvariety $\mathcal{S}$ of bounded residuated lattices that satisfies that for each $\mathbf{A} \in \mathcal{S}$ the application

$$\neg\neg : \mathbf{A} \to \mathbf{B}(\mathbf{A})$$

is a retraction onto the boolean skeleton.

Famous Stonean residuated lattices

- Boolean algebras
- Pseudocomplemented BL-algebras
- Product algebras
- Gödel algebras
- Pseudocomplemented MTL-algebras
- Stonean Heyting algebras

The category $\mathcal{T}$

Objects: Triples $(\mathbf{B}, \mathbf{D}, \phi)$ such that:

- $\mathbf{B}$ is a Boolean algebra,
- $\mathbf{D}$ is a residuated lattice and
- ϕ is bounded lattice-homomorphism,

$$\phi : \mathbf{B} \to F_i(\mathbf{D}).$$

The category $\mathcal{T}$

Morphisms: Given triples $(\mathbf{B}_i, \mathbf{D}_i, \phi_i)$, $i = 1, 2$, a morphism is a pair

$$(h, k) : (\mathbf{B}_1, \mathbf{D}_1, \phi_1) \to (\mathbf{B}_2, \mathbf{D}_2, \phi_2)$$

is a pair such that:

1. $h : \mathbf{B}_1 \to \mathbf{B}_2$ is a Boolean algebra homomorphism,
2. $k : \mathbf{D}_1 \to \mathbf{D}_2$ is a residuated lattice homomorphism, and
3. For all $a \in B_1$, $k(\phi_1(a)) \subseteq \phi_2(h(a))$.

The functor $\mathbf{T}$

$$\mathbf{T} : \mathcal{S} \to \mathcal{T}$$

$$\mathbf{A} \quad \mapsto \quad (\mathbf{B}(\mathbf{A}), \mathbf{D}(\mathbf{A}), \phi_A)$$

$$f : \mathbf{A}_1 \to \mathbf{A}_2 \quad \mapsto \quad \begin{array}{l} h : \mathbf{B}(\mathbf{A}_1) \to \mathbf{B}(\mathbf{A}_2) \\ k : \mathbf{D}(\mathbf{A}_1) \to \mathbf{D}(\mathbf{A}_2) \end{array}$$

Applications

Let $\mathbf{A} \in \mathcal{S}$:

congruences on $\mathbf{A}$	$\longleftrightarrow$	congruences $\mathbf{B}(\mathbf{A})$ and on $\mathbf{D}(\mathbf{A})$
duality on $\mathcal{S}$	$\longleftrightarrow$	dualities for the algebras that form the trip
categorical constructions on $\mathcal{S}$	$\longleftrightarrow$	categorical constructions on the algebras t

Thanks for your attention